

On a Hierarchy of Reflection Principles in Peano Arithmetic

Elena Nogina*

BMCC CUNY, Department of Mathematics

199 Chambers Street, New York, NY 10007

E.Nogina@gmail.com

Abstract

We study reflection principles of Peano Arithmetic PA which are based on both proof and provability. Any such reflection principle in PA is equivalent to either $\Box P \rightarrow P$ ($\Box P$ stands for *P is provable*) or $\Box^k u:P \rightarrow P$ for some $k \geq 0$ ($t:P$ states *t is a proof of P*). Reflection principles constitute a non-collapsing hierarchy with respect to their deductive strength

$$u:P \rightarrow P \prec \Box u:P \rightarrow P \prec \Box^2 u:P \rightarrow P \prec \dots \prec \Box P \rightarrow P.$$

1 Introduction

Reflection Principles are classical objects in Proof Theory. They were introduced by Rosser [18] and Turing [23] in the 1930s, and later studied by Feferman [8, 9], Kreisel and Lévi [11], Schmerl [19], Artemov [1], Beklemishev [5, 6], and many others (cf. survey [4]).

A *proof predicate* is a provably decidable formula *Proof* that enumerates all theorems of PA,

$$\text{PA} \vdash \varphi \quad \text{iff} \quad \text{Proof}(k, \varphi) \text{ for some } k.$$

*Supported by PSC CUNY Research Awards program.

In this paper all proof predicates are assumed *normal* ([3]), t.e.

1. for every k set

$$T(k) = \{\varphi \mid \text{Proof}(k, \varphi)\}$$

is finite, the function from k to $T(k)$ is computable;

2. for any k and l there is n such that

$$T(k) \cup T(l) \subseteq T(n).$$

Prime example: Gödel's proof predicate.

A natural example of a Reflection Principle is given by so-called local (or implicit) reflection. Let *Provable* F be $\exists x \text{Proof}(x, F)$. In the formal provability setting, the local reflection principle is the set of all arithmetical formulas

$$\text{Provable } F \rightarrow F,$$

where F is an arithmetical formula. Though all the instances of this reflection principle are true in the standard model of Peano Arithmetic **PA**, some of them are not provable. For example, if F is falsum \perp , the local reflection principle becomes Gödel's consistency formula

$$\neg \text{Provable } \perp.$$

Another example is given by the explicit reflection principle, i.e., the set of formulas

$$\text{Proof}(t, F) \rightarrow F$$

where t is an arbitrary proof term, and F an arithmetical formula. Here the situation is quite different; all instances of explicit reflection are provable.

Indeed, if $\text{Proof}(t, F)$ holds, then F is obviously provable in **PA**, and so is formula $\text{Proof}(t, F) \rightarrow F$. If $\neg \text{Proof}(t, F)$ holds, then it is provable in **PA** (since $\neg \text{Proof}(x, y)$ is decidable) and $\text{Proof}(t, F) \rightarrow F$ is again provable.

We study (cf. [17]) reflection principles of Peano Arithmetic **PA** which are based on both proof and provability predicates. (cf. [3, 7]).

Let P be a propositional letter and each of Q_1, Q_2, \dots, Q_m is either ' \square ' standing for provability in **PA**, or ' $u:$ ' standing for

$$'u \text{ is a proof of } \dots \text{ in } \mathbf{PA}',$$

u is a fresh proof variable. Then the formula

$$Q_1 Q_2 \dots Q_m P \rightarrow P$$

is called *generator*, and the set of all its arithmetical instances is the *reflection principle* corresponding to this generator. We will refer to reflection principles using their generators.

It is immediate that all reflection principles without explicit proofs ($Q_i = \Box$ for all i) are equivalent to the local reflection principle $\Box P \rightarrow P$. All \Box -free reflection principles are provable in PA and hence equivalent to $u:P \rightarrow P$. Mixing explicit proofs and provability yields infinitely many new reflection principles:

1. *Any reflection principle in PA is equivalent to either $\Box P \rightarrow P$ or $\Box^k u:P \rightarrow P$ for some $k \geq 0$.*
2. *Reflection principles constitute a non-collapsing hierarchy with respect to their deductive strength*

$$u:P \rightarrow P \prec \Box u:P \rightarrow P \prec \Box^2 u:P \rightarrow P \prec \dots \prec \Box P \rightarrow P.$$

The proofs essentially rely on introduced by the author Gödel-Löb-Artëmov logic GLA of formal provability and explicit proofs.

2 Description and basic properties of GLA

We describe the logic GLA introduced in [12] (see also [16]) in the union of the original languages of Gödel-Löb Logic GL(cf. [7, 21]) and Artemov's Logic of Proofs LP([3]).

The following two systems were predecessors of GLA:

- system B from [2], which did not have operations on proofs;
- system LPP from [20, 22] in an extension of languages of the logic of formal provability GL and the Logic of Proofs LP.

The immediate successors of GLA are the logic GrzA ([14]) of strong provability and explicit proofs and symmetric logic of proofs and provability ([15]).

Language of GLA.

Proof terms are built from *proof variables* x, y, z, \dots and *proof constants* a, b, c, \dots by means of two binary operations: *application* ‘ \cdot ’ and *union* ‘ $+$ ’, and one unary *proof checker* ‘ $!$ ’.

Formulas of GLA are defined by the grammar

$$A = S \mid A \rightarrow A \mid A \wedge A \mid A \vee A \mid \neg A \mid \Box A \mid t.A ,$$

where t stands for any proof term and S for any sentence letter.

Axioms and rules of both Gödel-Löb logic GL and LP, together with three specific principles connecting explicit proofs with formal provability, constitute GLA_\emptyset .

I. Axioms of classical propositional logic

Standard axioms of the classical logic (e.g., A1-A10 from [10])

II. Axioms of Provability Logic GL

$$\begin{array}{ll} \text{GL1 } \Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G) & \text{Deductive Closure/Normality} \\ \text{GL2 } \Box F \rightarrow \Box \Box F & \text{Positive Introspection/Transitivity} \\ \text{GL3 } \Box(\Box F \rightarrow F) \rightarrow \Box F & \text{Löb Principle} \end{array}$$

III. Axioms of the Logic of Proofs LP

$$\begin{array}{ll} \text{LP1 } s:(F \rightarrow G) \rightarrow (t:F \rightarrow [s \cdot t]:G) & \text{Application} \\ \text{LP2 } t:F \rightarrow !t:(t:F) & \text{Proof Checker} \\ \text{LP3 } s:F \rightarrow [s+t]:F, \quad t:F \rightarrow [s+t]:F & \text{Sum} \\ \text{LP4 } t:F \rightarrow F & \text{Explicit Reflection} \end{array}$$

IV. Axioms connecting explicit and formal provability

$$\begin{array}{ll} \text{C1 } t:F \rightarrow \Box F & \text{Explicit-Implicit connection} \\ \text{C2 } \neg t:F \rightarrow \Box \neg t:F & \text{Explicit-Implicit Negative Introspection} \\ \text{C3 } t:\Box F \rightarrow F & \text{Explicit-Implicit Reflection} \end{array}$$

V. Rules of inference

$$\begin{array}{ll} \text{R1 } F \rightarrow G, F \vdash G & \text{Modus Ponens} \\ \text{R2 } \vdash F \Rightarrow \vdash \Box F & \text{Necessitation} \\ \text{R3 } \vdash \Box F \Rightarrow \vdash F & \text{Reflection Rule} \end{array}$$

A **Constant Specification** CS for GLA is the set of formulas

$$\{c_1:A_1, c_2:A_2, c_3:A_3, \dots\},$$

where each A_i is an axiom of GLA_\emptyset and each c_i is a proof constant.

$$\text{GLA}_{CS} = \text{GLA}_\emptyset + CS,$$

$\text{GLA} = \text{GLA}_{CS}$ with the “total” CS .

One of the principal properties of GLA is its ability to internalize its own proofs [16]: *If $\text{GLA} \vdash F$, then for some proof term p , $\text{GLA} \vdash p:F$.*

An arithmetical interpretation $*$ of a GLA -formula is the direct sum of corresponding arithmetical interpretations for GL and LP ; in particular,

$$\begin{aligned} (\Box G)^* &= \text{Provable } G^*; \\ (p:F)^* &= \text{Proof}(p^*, F^*). \end{aligned}$$

GLA is sound with respect to the arithmetical provability interpretation ([12, 16]):

For any Constant Specification CS and any arithmetical interpretation $$ respecting CS , if $\text{GLA}_{CS} \vdash F$ then $\text{PA} \vdash F^*$.*

The following arithmetical completeness theorem holds ([12, 16]):

For any finite constant specification CS , if $\text{GLA}_{CS} \not\vdash F$, then for some interpretation $$ respecting CS , $\text{PA} \not\vdash F^*$.*

In [13, 16], GLA was supplied with Kripke-style semantics and found to be complete with respect to it.

3 Reflection principles in Peano Arithmetic

Fix a normal proof predicate Proof and, therefore, the corresponding provability predicate Provable . If F is a GLA -formula, then $\{F^*\}$ denotes the set of all arithmetical interpretations of F based on Proof and Provable .

Definition 1 Let P be a propositional letter and each of Q_1, Q_2, \dots, Q_m be either \Box or ‘ u .’ for some fresh proof variable u . Then a formula

$$Q_1 Q_2 \dots Q_m P \rightarrow P$$

is called a *generator* and the set $\{[Q_1 Q_2 \dots Q_m P \rightarrow P]^*\}$ is a *reflection principle* corresponding to this generator.

For example, the implicit reflection principle is generated by GLA -formula $\Box P \rightarrow P$, the explicit reflection is generated by $u:P \rightarrow P$.

Definition 2 Let G and H be GLA -formulas. We say that $\{H^*\} \preceq \{G^*\}$, or $H \preceq G$, for short, if $\text{PA} + \{G^*\}$ proves all formulas from $\{H^*\}$. $H \simeq G$ (is read as “ H is equivalent to G ”) means that both $H \preceq G$ and $G \preceq H$ hold; $H \prec G$ stands for ($H \preceq G$ and $H \not\simeq G$).

Example:

$$u:P \rightarrow P \prec \Box P \rightarrow P.$$

We study the structure of reflection principles in the explicit-implicit language. In particular, we establish classification of reflection principles (Theorem 4):

Any reflection principle is equivalent to either $\Box P \rightarrow P$ or, for some $k \geq 0$, to $\Box^k u:P \rightarrow P$.

We also discover that reflection principles constitute a hierarchy (Theorem 5):

$$u:P \rightarrow P \prec \Box u:P \rightarrow P \prec \Box^2 u:P \rightarrow P \prec \dots \prec \Box P \rightarrow P.$$

These two results could be immediately concluded from the well-known fact (Lemma 2):

$$\neg \perp \prec \neg \Box \perp \prec \neg \Box^2 \perp \prec \dots \prec \Box P \rightarrow P$$

together with the following assertions we will establish in this section:

1. For each $n \geq 1$, $\Box^n P \rightarrow P \simeq \Box P \rightarrow P$ (Uniqueness of Provability Reflection, Theorem 1);
2. For $k \geq 0$, $\Box^k u:Q_1 Q_2 \dots Q_n P \rightarrow P \simeq \Box^k u:P \rightarrow P$ (Theorem 3);
3. For $k \geq 0$, $\Box^k u:P \rightarrow P \simeq \neg \Box^k \perp$, (Theorem 6).

3.1 Uniqueness of Provability Reflection

Let $Q_1 Q_2 \dots Q_m P \rightarrow P$ be a generator, and $Q_1 Q_2 \dots Q_m$ consists only of implicit provability operators \Box . It is obvious that the corresponding principle is equivalent to $\Box P \rightarrow P$.

Theorem 1 (Uniqueness of Provability Reflection)

Proof. In light of the arithmetic soundness of \mathbf{GLA}_\emptyset ,

$$\Box P \rightarrow P \preceq \Box^n P \rightarrow P$$

follows from the fact that $\mathbf{GLA}_\emptyset \vdash \Box P \rightarrow \Box^n P$. The converse inequality $\Box^n P \rightarrow P \preceq \Box P \rightarrow P$ is implied by the fact that

$$[\Box^n P \rightarrow \Box^{n-1} P] \wedge [\Box^{n-1} P \rightarrow \Box^{n-2} P] \wedge \dots \wedge [\Box P \rightarrow P] \rightarrow [\Box^n P \rightarrow P]$$

is derivable in \mathbf{GLA}_\emptyset . □

3.2 Leading-Explicit Reflection Principles are provable

Theorem 2 For any $n \geq 0$, $\text{GLA}_\emptyset \vdash u:Q_1Q_2 \dots Q_nP \rightarrow P$.

Proof. Induction on n . The base case $n = 0$ is trivial. For the induction step consider two cases.

Case 1: Q_1 is “ v ” for some proof variable v . Then, by explicit reflection,

$$\text{GLA}_\emptyset \vdash u:Q_1Q_2 \dots Q_nP \rightarrow v:Q_2 \dots Q_nP.$$

By the Induction Hypothesis,

$$\text{GLA}_\emptyset \vdash v:Q_2 \dots Q_nP \rightarrow P.$$

Hence

$$\text{GLA}_\emptyset \vdash u:Q_1Q_2 \dots Q_nP \rightarrow P.$$

Case 2: Q_1 is \Box . Then $u:Q_1Q_2 \dots Q_nP$ has type $u:\Box^m F$ for some $m \geq 1$, where F is either P or $w:Q_{n-m-1} \dots Q_nP$. Now we show that $\text{GLA}_\emptyset \vdash u:\Box^m F \rightarrow F$. Indeed,

- | | |
|--|-----------------------------|
| 1. $\neg\Box^m F \rightarrow \neg u:\Box^m F$, | by E-reflection; |
| 2. $\neg u:\Box^m F \rightarrow \Box(\neg u:\Box^m F)$, | axiom C2 ; |
| 3. $\Box(\neg u:\Box^m F) \rightarrow \Box(u:\Box^m F \rightarrow F)$, | by reasoning in GL ; |
| 4. $\neg\Box^m F \rightarrow \Box(u:\Box^m F \rightarrow F)$, | from 1,2, and 3; |
| 5. $\Box(u:\Box^m F \rightarrow F) \rightarrow \Box^m(u:\Box^m F \rightarrow F)$, | from transitivity; |
| 6. $\neg\Box^m F \rightarrow \Box^m(u:\Box^m F \rightarrow F)$, | from 4 and 5; |
| 7. $\Box^m F \rightarrow \Box^m(u:\Box^m F \rightarrow F)$, | by reasoning in GL ; |
| 8. $\Box^m(u:\Box^m F \rightarrow F)$, | from 6 and 7; |
| 9. $u:\Box^m F \rightarrow F$, | by Reflection Rule. |

If F is P we are done; if F is $w:Q_{n-m-1} \dots Q_nP$, then, by the Induction Hypothesis, $\text{GLA}_\emptyset \vdash F \rightarrow P$ which yields the theorem claim as well. \square

Corollary 1 (Uniqueness of Leading-Explicit Reflection) Let $u:Q_1Q_2 \dots Q_nP \rightarrow P$ be a reflection principle generator. Then

$$u:Q_1Q_2 \dots Q_nP \rightarrow P \simeq u:P \rightarrow P.$$

Proof. Follows from Theorem 2 by the arithmetical soundness of GLA_\emptyset . \square

3.3 Classification of Reflection Principles

Theorem 3 *Let $k \geq 0$ and $\Box^k u:Q_1 Q_2 \dots Q_n P \rightarrow P$ be a reflection principle generator. Then*

$$\Box^k u:Q_1 Q_2 \dots Q_n P \rightarrow P \simeq \Box^k u:P \rightarrow P.$$

Proof. The following argument could not be done in GLA; so, we reason in PA instead.

First, we establish “ \preceq ”, i.e.,

$$\text{PA}' = \text{PA} + \{[\Box^k u:P \rightarrow P]^*\} \vdash \{[\Box^k u:Q_1 Q_2 \dots Q_n P \rightarrow P]^*\}.$$

Fix an interpretation $*$. By Theorem 1,

$$\text{PA} \vdash u^*:[Q_1 Q_2 \dots Q_n P]^* \rightarrow P^*.$$

We write $t:F$ for *Proof*(t, F) and $\Box F$ for *Provable* F in PA, for brevity.

Let s be its proof in PA. Then,

$$\text{PA} \vdash s:(u^*:[Q_1 Q_2 \dots Q_n P]^* \rightarrow P^*).$$

By proof checking and internalized *Modus Ponens* in PA, we can find an arithmetical proof t such that

$$\text{PA} \vdash u^*:[Q_1 Q_2 \dots Q_n P]^* \rightarrow t:P^*,$$

from which we conclude

$$\text{PA} \vdash \Box^k u^*:[Q_1 Q_2 \dots Q_n P]^* \rightarrow \Box^k t:P^*,$$

$$\text{PA}' \vdash \Box^k u^*:[Q_1 Q_2 \dots Q_n P]^* \rightarrow P^*,$$

i.e.,

$$\text{PA}' \vdash [\Box^k u:Q_1 Q_2 \dots Q_n P \rightarrow P]^*.$$

Let us now establish “ \succeq ”, i.e., that

$$\text{PA}'' = \text{PA} + \{[\Box^k u:Q_1 Q_2 \dots Q_n P \rightarrow P]^*\} \vdash \{[\Box^k u:P \rightarrow P]^*\}.$$

Lemma 1 *For each interpretation $*$ there is an interpretation \sharp which coincides with $*$ on P such that*

$$\text{PA} \vdash [u:P]^* \rightarrow [u:Q_1 Q_2 \dots Q_n P]^\sharp.$$

Proof. By induction on n . The case $n = 0$ is trivial. Let for some interpretation \flat coinciding with $*$ on P ,

$$\text{PA} \vdash [u:P]^* \rightarrow [u:Q_2 \dots Q_n P]^\flat.$$

By proof-checking,

$$\text{PA} \vdash [u:P]^* \rightarrow !u^\flat u^\flat : [Q_2 \dots Q_n P]^\flat.$$

Case 1. If Q_1 is a proof variable v , then define u^\sharp as $!u^\flat$, v^\sharp as u^\flat , set \sharp to be \flat everywhere else, and get the desired

$$\text{PA} \vdash [u:P]^* \rightarrow [u:v:Q_2 \dots Q_n P]^\sharp.$$

Case 2. If Q_1 is \Box , then by reasoning in PA find a proof t such that

$$\text{PA} \vdash !u^\flat u^\flat : [Q_2 \dots Q_n P]^\flat \rightarrow t : \Box [Q_2 \dots Q_n P]^\flat,$$

therefore, $\text{PA} \vdash [u:P]^* \rightarrow t : \Box [Q_2 \dots Q_n P]^\flat$. Define $u^\sharp = t$ (u is fresh!) and set \sharp equal \flat everywhere else. Then

$$\text{PA} \vdash [u:P]^* \rightarrow [u:Q_1 Q_2 \dots Q_n P]^\sharp,$$

which completes theorem's proof. \square

Now, by the standard PA -reasoning,

$$\text{PA} \vdash [\Box^k u:P]^* \rightarrow [\Box^k u:Q_1 Q_2 \dots Q_n P]^\sharp,$$

and since

$$\text{PA}'' \vdash [\Box^k u:Q_1 Q_2 \dots Q_n P \rightarrow P]^\sharp$$

and $P^\sharp = P^*$ we conclude that

$$\text{PA}'' \vdash [\Box^k u:P \rightarrow P]^*.$$

\square

From Theorems 1 and 3 immediately follows

Theorem 4 (Classification of Reflection Principles) *Any reflection principle is equivalent to either*

$$\Box P \rightarrow P$$

or, for some $k \geq 0$, to

$$\Box^k u:P \rightarrow P.$$

Proof. Consider an arbitrary reflection principle π

$$Q_1 Q_2 \dots Q_n P \rightarrow P.$$

If all Q_i are \Box 's, then, by Theorem 1, $\pi = \Box P \rightarrow P$. Otherwise, π can be written as

$$\Box^k u:Q_{n-k-1} \dots Q_n P \rightarrow P$$

for an appropriate $k \geq 0$. By Theorem 3, $\pi = \Box^k u:P \rightarrow P$. \square

3.4 Hierarchy of Reflection Principles

Theorem 5 *Reflection principles form a linear ordering*

$$u:P \rightarrow P \prec \Box u:P \rightarrow P \prec \Box^2 u:P \rightarrow P \prec \dots \prec \Box P \rightarrow P.$$

This Theorem is an immediate corollary of the following two assertions.

Theorem 6 *For each $k \geq 0$, $\Box^k u:P \rightarrow P \simeq \neg \Box^k \perp$.*

Proof. Putting $P = \perp$ we get $\Box^k u:P \rightarrow P \succeq \neg \Box^k \perp$. For the converse, argue in GLA_\emptyset . Case $k = 0$ is trivial. Let $k \geq 1$. Assume $\neg \Box^k \perp$, $\Box^k u:P$, and $\neg P$ and look for a contradiction. By explicit reflection, from $\neg P$ we derive $\neg u:P$ and, by explicit-implicit negative introspection, $\Box \neg u:P$. By transitivity, we get $\Box^k \neg u:P$. From this and $\Box^k u:P$, by the usual modal reasoning we conclude $\Box^k (\neg u:P \wedge u:P)$; hence $\Box^k \perp$, a contradiction. \square

Now, to get Theorem 5, it suffices to refer to a well-known fact:

Lemma 2

$$\neg \perp \prec \neg \Box \perp \prec \neg \Box^2 \perp \prec \neg \Box^3 \perp \prec \dots \prec \Box P \rightarrow P.$$

Proof.

a) For $k \geq 1$, by transitivity, $\mathbf{GL} \vdash \Box^{k-1}\perp \rightarrow \Box^k\perp$, hence $\mathbf{GL} \vdash \neg\Box^k\perp \rightarrow \neg\Box^{k-1}\perp$. By the arithmetical soundness of \mathbf{GL} ,

$$\neg\Box^{k-1}\perp \preceq \neg\Box^k\perp.$$

Modal formula $\Box^k\perp \rightarrow \Box^{k-1}\perp$ is false at the root of a k -node linear model, hence not provable in \mathbf{GL} . By the arithmetical completeness of \mathbf{GL} , $\mathbf{PA} \nvdash \Box^k\perp \rightarrow \Box^{k-1}\perp$, hence

$$\neg\Box^k\perp \not\preceq \neg\Box^{k-1}\perp,$$

therefore

$$\neg\Box^{k-1}\perp \prec \neg\Box^k\perp.$$

b) For each $k \geq 0$, $\neg\Box^k\perp \preceq \Box P \rightarrow P$. Indeed, cases of $k = 0, 1$ are trivial. Consider $k \geq 2$. From instances of $\Box P \rightarrow P$

$$\Box^k\perp \rightarrow \Box^{k-1}\perp, \Box^{k-1}\perp \rightarrow \Box^{k-2}\perp, \dots, \Box\perp \rightarrow \perp,$$

by a chain of syllogisms, we derive $\Box^k\perp \rightarrow \perp$, and hence

$$\mathbf{PA} + \{[\Box P \rightarrow P]^*\} \vdash \neg\Box^k\perp.$$

c) For any $k \geq 0$, $\Box P \rightarrow P \not\preceq \neg\Box^k\perp$. Suppose the opposite, namely, that for some $k \geq 0$, $\Box P \rightarrow P \preceq \neg\Box^k\perp$. Since, by b),

$$\neg\Box^{k+1}\perp \preceq \Box P \rightarrow P,$$

we have $\neg\Box^{k+1}\perp \preceq \neg\Box^k\perp$, which is impossible, by a). □

4 Acknowledgements

The author is grateful to Sergei Artemov, Melvyn Fitting, Hidenori Kurokawa, Anil Nerode, Junhua Yu and logic groups of National Chung Cheng University, Academia Sinica and the Computational Logic seminar of the CUNY Graduate Center for useful discussions.

References

- [1] S. Artemov. Arithmetically complete modal theories. Russian, English translation in: *Amer. Math. Soc. Transl 2*, 135: 39–54, 1987
- [2] S. Artemov. Logic of proofs. *Annals of Pure and Applied Logic*, 67(1):29–59, 1994.
- [3] S. Artemov. Explicit provability and constructive semantics. *Bulletin of Symbolic Logic*, 7(1):1–36, 2001.
- [4] S. Artemov and L. Beklemishev. Provability logic. In D. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic, 2nd ed.*, volume 13, pages 189–360. Springer, Dordrecht, 2005.
- [5] L. Beklemishev. Induction rules, reflection principles, and provably recursive functions. *Annals of Pure and Applied Logic*, 85(3):193–242, 1997.
- [6] L. Beklemishev. Proof-theoretic analysis by iterated reflection. *Archive for Mathematical Logic*, 42(6):515–552, 2003.
- [7] G. Boolos. *The Logic of Provability*. Cambridge University Press, Cambridge, 1993.
- [8] S. Feferman. Arithmetization of metamathematics in a general setting. *Fundamenta Mathematicae*, 49:35–92, 1960.
- [9] S. Feferman. Transfinite recursive progressions of axiomatic theories. *The Journal of Symbolic Logic*, 27:259–316, 1962.
- [10] S. Kleene. *Introduction to Metamathematics*. Van Norstrand, 1952.
- [11] G. Kreisel and A. Lévy. Reflection Principles and their Use for Establishing the Complexity of Axiomatic Systems. *Mathematical Logic Quarterly* 14 (7-12):97–142, 1968.
- [12] E. Nogina. On logic of proofs and provability. *Bulletin of Symbolic Logic*, 12(2):356, 2006.
- [13] E. Nogina. Epistemic completeness of GLA. *Bulletin of Symbolic Logic*, 13(3):407, 2007.

- [14] E. Nogina. Logic of Strong Provability and Explicit Proofs. *Bulletin of Symbolic Logic*, 15(1):124–125, 2009.
- [15] E. Nogina. Symmetric Logic of Proofs and Provability. 2010 Spring AMS Eastern Sectional Meeting May 22-23, 2010 New Jersey Institute of Technology, Newark, NJ, 2010.
<http://www.ams.org/meetings/sectional/1060-03-29.pdf>
- [16] E. Nogina. On Logic of Formal Provability and Explicit Proofs. *ArXiv*, 2014.
- [17] E. Nogina. On Explicit-Implicit Reflection Principles. To appear in *Bulletin of Symbolic Logic* 2014.
- [18] B. Rosser. Extensions of Some Theorems of Gödel and Church. *The Journal of Symbolic Logic*, 1(3):87–91, 1936.
- [19] U.R. Schmerl. A fine structure generated by reflection formulas over Primitive Recursive Arithmetic. *Studies in Logic and the Foundations of Mathematics* 97: 335-350, 1979
- [20] T. Sidon. Provability logic with operations on proofs. In S. Adian and A. Nerode, editors, *Logical Foundations of Computer Science' 97, Yaroslavl'*, volume 1234 of *Lecture Notes in Computer Science*, pages 342–353. Springer, 1997.
- [21] R.M. Solovay. Provability interpretations of modal logic. *Israel Journal of Mathematics*, 28:33–71, 1976.
- [22] T. Yavorskaya (Sidon). Logic of proofs and provability. *Annals of Pure and Applied Logic*, 113(1-3):345–372, 2002.
- [23] A.M. Turing. Systems of logic based on ordinals. *Proceedings of the London Mathematical Society* 2(1):161–228, 1939